Bhabha relativistic wave equations

1997 J. Phys. A: Math. Gen. 30 4005

(http://iopscience.iop.org/0305-4470/30/11/027)

J. Phys. A: Math. Gen. **30** (1997) 4005–4017. Printed in the UK PII: S0305-4470(97)80146-1

Bhabha relativistic wave equations

R-K Loide†, I Ots‡ and R Saar§

† Tallinn Technical University, Ehitajate tee 5, EE0026 Tallinn, Estonia
‡ Institute of Physics, Riia 142, EE2400 Tartu, Estonia
§ Estonian Agricultural University, Kreutzwaldi 56, EE2400 Tartu, Estonia

Received 29 November 1996

Abstract. It is a well known fact that the Dirac and Kemmer–Duffin equations are the Bhabha equations. We use the method based on the de Sitter group $SO(1,4)$ to show that the Rarita–Schwinger and Bargmann–Wigner equations can also be treated as the Bhabha equations with some subsidiary conditions. This demonstrates that the de Sitter group can be considered as a significant auxiliary group which provides a unified approach to the equations of relativistic quantum theory.

1. Introduction

More than 50 years ago Bhabha wrote a paper 'Relativistic wave equations for the elementary particles', where a new class of relativistic wave equations was proposed (Bhabha 1945). Bhabha introduced a class of multiparticle equations which appeared to be related to the de Sitter group $SO(1,4)$. The same equations were previously introduced and analysed by Lubanski (1942a, b), but in the current physical literature the equations connected with the de Sitter group are known as Bhabha equations.

Until the Lubanski and Bhabha works different single particle equations were proposed and investigated (Dirac 1936, Duffin 1938, Kemmer 1939, Fierz 1939, Rarita and Schwinger 1941). Equations proposed by Lubanski and Bhabha were multiparticle equations. It is interesting to note that the theory of multiparticle higher-spin equations is still actually due to the presence of difficulties in higher-spin single particle theories in the presence of interactions (Velo and Zwanziger 1969a, b). Several attempts have been made to overcome these difficulties, however, the results obtained have yet to solve the problem (for our efforts on the subject see, for example, Saar *et al* 1993, Loide *et al* 1992, 1994, Saar *et al* 1994). In multiparticle theories similar acausality difficulties may be avoided. Moreover, different new field theoretical models, such as string and supersymmetrical models reduce to the investigation of higher-spin fields and wave equations.

In the present paper we take another look at the relativistic wave equations proposed by Bhabha and Lubanski, and discuss their relation with the de Sitter group. It should be mentioned that Bhabha equations were widely discussed in the 1970s (Krajcik and Nieto 1974, 1975, 1976a, b, 1977). Our treatment here is somewhat different from the previous ones and is based mainly on the de Sitter group (Kõiv 1969, Kõiv *et al* 1970, Loide 1972, Kõiv and Saar 1974, Loide 1975). We demonstrate that the de Sitter group should be treated as an important auxiliary group providing a unified approach to the standard equations of relativistic quantum field theory.

0305-4470/97/114005+13$19.50

We discuss the general relativistic wave equation

$$(\mathrm{i}\partial_\mu \beta^\mu - m)\psi(x) = 0 \tag{1.1}$$

where $\psi(x)$ is some finite-dimensional Lorentz field and the Lorentz generators $S^{\mu\nu}$ are expressed via the β^μ matrices in the following form:

$$S^{\mu\nu} = k^2[\beta^\mu, \beta^\nu] \tag{1.2}$$

where k is some numerical coefficient.

The general theory of equation (1.1) was rather well developed in the 1940s and it has been discussed in many textbooks (Corson 1952, Gelfand *et al* 1963). However, the application of the general theory to higher-spin particles is not simple and therefore the higher-spin wave equations proposed so far are not always correct (Loide 1984, 1985). For this reason some new methods have been proposed which have allowed us to deal with higher-spin wave equations. In this paper we shall discuss one important special class of equation (1.1)—the Bhabha relativistic wave equations.

The most important feature of the equations satisfying (1.2) is a very close relation with the de Sitter group $SO(1, 4)$. Indeed, if we introduce the operators

$$S^{\mu 5} = k\beta^\mu \tag{1.3}$$

the generators $S^{\mu 5}$ and $S^{\mu\nu}$ satisfy the commutation relations of the de Sitter algebra:

$$\begin{aligned}
&[S^{\mu 5}, S^{\nu 5}] = S^{\mu\nu} \\
&[S^{\mu\nu}, S^{\rho 5}] = g^{\nu\rho} S^{\mu 5} - g^{\mu\rho} S^{\nu 5} \\
&[S^{\mu\nu}, S^{\rho\sigma}] = g^{\nu\rho} S^{\mu\sigma} + g^{\mu\sigma} S^{\nu\rho} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho}.
\end{aligned} \tag{1.4}$$

As we shall see later, the wavefunction $\psi(x)$ is connected with the representations of the de Sitter group $SO(1, 4)$. We consider only finite-dimensional representations and demonstrate that besides the Dirac and the Kemmer–Duffin equations which are the Bhabha equations, also the Rarita–Schwinger and the Bargmann–Wigner equations can be given in the Bhabha form with some subsidiary conditions. The corresponding functions $\psi(x)$ are finite-dimensional representations of the de Sitter group.

It is interesting to note that in physics the de Sitter group plays quite an important role, since in many problems one must use the de Sitter group and its representations, in some cases in a four-dimensional, in some cases in a five-dimensional form. Here we note some important mathematical results, which connect Poincaré and de Sitter groups—in the theory of contractions and deformations of groups and algebras it has been proved that contraction of the de Sitter group is the Poincaré group (Inonü and Wigner 1953) and from deformations of the Poincaré group one obtains the de Sitter group (Levy-Nahas 1967, Lyakhovsky 1969). From these mathematical results it follows that two groups—the Poincaré group, which is the most important symmetry group in physics, and the de Sitter group are very closely related. The physical origin of such relations is not yet clear, but we hope it will get some clarification in the future.

The paper is organized as follows. In section 2, the general structure of Bhabha equations is clarified. In sections 3 and 4, the Rarita–Schwinger equation and its Bhabha structure is studied. In section 5, the Bargmann–Wigner equation is discussed.

2. Mass and spin of the Bhabha equations

We start with a few words about finite-dimensional representations of the de Sitter group (see, for example, Antoine and Speiser, 1964a, b). Each finite-dimensional representation

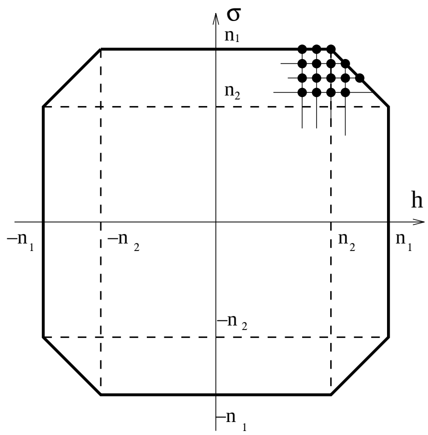

Figure 1. Weight diagram for the representation (n_1, n_2).

can be characterized by two integers or half-odd integers (n_1, n_2), satisfying $n_1 \geqslant n_2$. To each representation (n_1, n_2) corresponds the octagonal weight diagram (figure 1).

The weights (h, σ) are determined from the eigenvalue problem of Cartan subalgebra:

$$\begin{aligned} S^{05}\psi &= h\psi \\ \mathrm{i}S^{12}\psi &= \sigma\psi. \end{aligned} \tag{2.1}$$

The components of each weight $(h, \sigma) - h$ and σ may both have values $n_1, n_1 - 1, \ldots, -n_1 + 1, -n_1$, but only such values for h and σ are allowed, which give the weight point inside the octagonal weight diagram. The edge points of the weight diagram are single, the inner points are in general multiple. The multiplicity of weight points can be calculated (Kõiv 1967), but for lower representations the multiplicity of weight points can be found from the simple symmetry considerations of the weight diagram—the weight points (h, σ) and (σ, h) have the same multiplicity.

Next we apply the results to the Bhabha equation (1.1) corresponding to some irreducible representation (n_1, n_2) of the de Sitter group. For simplicity we use further the momentum representation

$$(p_\mu \beta^\mu - m)\psi(p) = 0. \tag{2.2}$$

Equation (2.2), in general, describes states with different mass and spin. In order to establish the mass and spin spectrum of a given equation we introduce new operators

$$\begin{aligned} S^{\mu 5}(p) &= \epsilon^\mu{}_\nu S^{\nu 5} \\ S^{\mu\nu}(p) &= \epsilon^\mu{}_\rho \epsilon^\nu{}_\sigma S^{\rho\sigma} \end{aligned} \tag{2.3}$$

where

$$\epsilon^0{}_\mu = \frac{p_\mu}{\sqrt{p^2}}$$

and

$$g^{\sigma\rho}\epsilon^\mu{}_\rho \epsilon^\nu{}_\sigma = g^{\mu\nu}.$$

$\sqrt{p^2}$ is the mass of a given state.

It is easy to verify that the introduced generators also satisfy the commutation relations of the de Sitter group (1.4).

Since $S^{\mu 5} = k\beta^{\mu}$, one can write (2.2) as

$$(p_\mu S^{\mu 5} - km)\psi = 0 \tag{2.4}$$

and by using (2.3), it turns into the eigenvalue problem

$$S^{05}(p)\psi = h\psi \tag{2.5}$$

where

$$h = \frac{km}{\sqrt{p^2}}. \tag{2.6}$$

The spin projection is determined from the eigenvalue problem

$$\mathrm{i}S^{12}(p)\psi = \sigma\psi. \tag{2.7}$$

In the case of a finite-dimensional irreducible representation (n_1, n_2) one can obtain the mass and spin projection spectrum. The mass of states is determined from

$$\sqrt{p^2} = \frac{km}{h}. \tag{2.8}$$

Naturally, $h = 0$ is excluded due to $m \neq 0$. In general,

$$\sqrt{p^2} = \frac{km}{n_1}, \frac{km}{n_1 - 1}, \ldots, \frac{-km}{n_1}$$
$$\sigma = n_1, n_1 - 1, \ldots, -n_1.$$

From the given results it follows that in general the Bhabha equation describes particles with several mass and spin. If one single state with a given mass and spin is needed, some auxiliary conditions must be applied.

From the well known equations, the Dirac equation and Kemmer–Duffin equations are Bhabha equations. The Dirac equation corresponds to the representation $(\frac{1}{2}, \frac{1}{2})$. Spin $\frac{1}{2}$ is described by the nonzero eigenvalues $h = \pm\frac{1}{2}$. The Kemmer–Duffin spin 0 equation corresponds to the representation $(1, 0)$, spin 0 is described by the nonzero eigenvalues $h = \pm 1$. The Kemmer–Duffin spin 1 equation corresponds to the representation $(1, 1)$ and now spin 1 is described by the eigenvalues $h = \pm 1$.

In the following sections we will show that the Rarita–Schwinger and the Bargmann–Wigner equations are also connected with the de Sitter group and can be treated as Bhabha equations with some auxiliary conditions.

3. Rarita–Schwinger spin $\frac{3}{2}$ equation

The Rarita–Schwinger equation for a particle with spin $s = \frac{3}{2}$ has a form (Rarita and Schwinger 1941)

$$(p_\mu \gamma^\mu - m)\psi^\alpha = 0 \tag{3.1}$$

with the additional condition

$$\gamma_\alpha \psi^\alpha = 0. \tag{3.2}$$

Here ψ^α is a vector-bispinor field. Since vector-bispinor is also the irreducible representation $(\frac{3}{2}, \frac{1}{2})$ of the de Sitter group, one can supplement the Lorentz generators $S^{\mu\nu}$ with generators $S^{\mu 5}$ generating the de Sitter algebra.

For a vector-bispinor the Lorentz generators $S^{\mu\nu}$ are

$$S^{\mu\nu} = I^{\mu\nu} \times E + I_4 \times \tfrac{1}{2}\gamma^\mu\gamma^\nu \tag{3.3}$$

where E and I_4 are unit operators acting on spinor and vector indices. The generators $I^{\mu\nu}$ are

$$(I^{\mu\nu})^{\alpha}{}_{\beta} = g^{\mu\alpha} g^{\nu}{}_{\beta} - g^{\nu\alpha} g^{\mu}{}_{\beta}. \tag{3.4}$$

de Sitter generators $S^{\mu 5}$ may be repsesented in the form

$$(S^{\mu 5})^{\alpha}{}_{\beta} = \tfrac{1}{2}[\gamma^{\mu} g^{\alpha}{}_{\beta} - (1+\mathrm{i}) g^{\mu\alpha}\gamma_{\beta} - (1-\mathrm{i})\gamma^{\alpha} g^{\mu}{}_{\beta} + \gamma^{\alpha}\gamma^{\mu}\gamma_{\beta}]. \tag{3.5}$$

Now we will prove that the Rarita–Schwinger equation is equivalent to the Bhabha equation

$$p_{\mu} S^{\mu 5}\psi = \frac{m}{2}\psi \tag{3.6}$$

with an additional condition (3.2). From figure 2 we can see that there are nonzero eigenvalues $h = \pm\frac{1}{2}$ and $h = \pm\frac{3}{2}$. The eigenvalues $h = \pm\frac{1}{2}$ describe states with spin $\frac{3}{2}$ and spin $\frac{1}{2}$. The mass of the corresponding states is equal to m. Eigenvalues $h = \pm\frac{3}{2}$ describe spin $\frac{1}{2}$ with the mass $m/3$.

Equation (3.6) is equivalent to the eigenvalue problem

$$S^{\mu 5}(p)\psi = h\psi \tag{3.7}$$

where

$$(S^{05}(p))^{\alpha}{}_{\beta} = \tfrac{1}{2}\Gamma^{0}(g^{\alpha}{}_{\beta} - \gamma^{\alpha}\gamma_{\beta}) + \tfrac{1}{2}(1-\mathrm{i})(\varepsilon^{0\alpha}\gamma_{\beta} - \gamma^{\alpha}\varepsilon^{0}{}_{\beta}) \tag{3.8}$$

and $\Gamma^{\mu} = \varepsilon^{\mu}{}_{\nu}\gamma^{\nu}$.

Let us consider the eigenfunction ψ which satisfies (3.2): $\gamma_{\alpha}\psi^{\alpha} = 0$. From

$$\varepsilon^{0}{}_{\alpha} S^{05}(p)^{\alpha}{}_{\beta} = \frac{\mathrm{i}}{2}(\varepsilon^{0}{}_{\beta}\Gamma^{0} - \gamma_{\beta})$$

we obtain

$$\frac{\mathrm{i}}{2}\Gamma^{0}(\varepsilon^{0}{}_{\alpha}\psi^{\alpha}) = h(\varepsilon^{0}{}_{\alpha}\psi^{\alpha}). \tag{3.9}$$

Since Γ^{μ} satisfy $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu}$, eigenvalues of Γ^{0} are ± 1. From (3.9) it therefore follows that $\varepsilon^{0}{}_{\alpha}\psi^{\alpha} = 0$, i.e.

$$p_{\mu}\psi^{\mu} = 0. \tag{3.10}$$

For the wavefunction ψ satisfying (3.2) and (3.10) from (3.7) we obtain that

$$S^{05}(p)^{\alpha}{}_{\beta}\psi^{\beta} = \tfrac{1}{2}\Gamma^{0}\psi^{\alpha}.$$

Consequently, conditions (3.2) and (3.9) discard the solutions with $h = \pm\frac{3}{2}$, equation (3.6) reduces to (3.1), describing single mass m.

Now it remains to prove that due to these conditions only spin 3/2 remains. The spin operator is as follows

$$\boldsymbol{S}(p) = \mathrm{i}(S^{23}(p), S^{31}(p), S^{12}(p)). \tag{3.11}$$

For $S^{2}(p)$ one can write

$$S^{2}(p) = \tfrac{3}{2}(\tfrac{3}{2}+1) + l_{1}(p) + l_{2}(p) \tag{3.12}$$

where

$$l_{1}(p)^{\alpha}{}_{\beta} = -(\gamma^{\alpha} - \varepsilon^{0\alpha}\Gamma^{0})\gamma_{\beta}$$

and

$$l_{2}(p)^{\alpha}{}_{\beta} = (\gamma^{\alpha}\Gamma^{0} - 4\varepsilon^{0\alpha})\varepsilon^{0}{}_{\beta}.$$

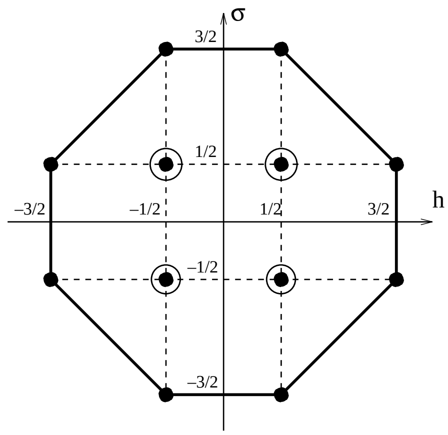

Figure 2. Weight diagram for the representation $(\frac{3}{2}, \frac{1}{2})$.

Now from $\gamma_\mu \psi^\mu = \varepsilon^0{}_\mu \psi^\mu = 0$ it follows that

$$S^2(p)\psi = \tfrac{3}{2}(\tfrac{3}{2}+1)\psi$$

and therefore the original Rarita–Schwinger spin $\frac{3}{2}$ equation is equivalent to the Bhabha equation corresponding to the irreducible representation $(\frac{3}{2}, \frac{1}{2})$ of the de Sitter group.

4. General Rarita–Schwinger equation

The Rarita–Schwinger equation for a particle with the spin $s = n + \frac{1}{2}$ has a form (Rarita and Schwinger 1941)

$$(p_\mu \gamma^\mu - m)\psi^{\alpha_1 \ldots \alpha_n} = 0 \tag{4.1}$$

with an additional condition

$$\gamma_\beta \psi^{\beta\alpha_2\ldots\alpha_n} = 0. \tag{4.2}$$

Since symmetrical tensor-bispinor $\psi^{\alpha_1\ldots\alpha_n}$ is also the representation $(n + \frac{1}{2}, \frac{1}{2})$ of the de Sitter group, one can show, similarly to the previous section, that equations (4.1) and (4.2) are equivalent to the Bhabha equation

$$p_\mu S^{\mu 5}\psi = \frac{m}{2}\psi$$

with auxiliary condition (4.2).

For symmetrical tensor-bispinor the Lorentz generators $S^{\mu\nu}$ are

$$S^{\mu\nu} = I^{\mu\nu} \times E + I_{4n} \times \tfrac{1}{2}\gamma^\mu\gamma^\nu \tag{4.3}$$

where $I^{\mu\nu}$ are the matrices

$$I^{\mu\nu} = \sum_{r=1}^{n} I_1 \times \cdots \times I_{r-1} \times I_r{}^{\mu\nu} \times I_{r+1} \times \cdots \times I_n \tag{4.4}$$

and the vector generators $I_r{}^{\mu\nu}$ are given in (3.4).

From the weight diagram of irreducible representation $(n + \frac{1}{2}, \frac{1}{2})$ (figure 3) one can see that the spin $s = n + \frac{1}{2}$ corresponds to the weight points $h = \pm\frac{1}{2}$. Naturally, there are more

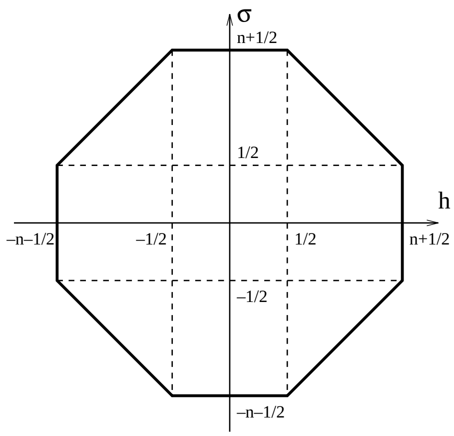

Figure 3. Weight diagram for the irreducible representation $(n+\frac{1}{2}, \frac{1}{2})$.

spins corresponding to these weight points, therefore we must prove that (4.2) separates points with $h = \pm\frac{1}{2}$ and extracts spin $s = n + \frac{1}{2}$.

At the beginning we find the Casimir operators of the Lorentz group

$$\begin{aligned} F(p) &= \tfrac{1}{4} S_{\mu\nu}(p) S^{\mu\nu}(p) \\ G(p) &= \tfrac{1}{8} \varepsilon_{\mu\nu\rho\sigma} S^{\mu\nu}(p) S^{\rho\sigma}(p). \end{aligned} \tag{4.5}$$

If we denote the eigenvalues of $F(p)$ and $G(p)$ corresponding to the Lorentz group representations $((n+1)/2, n/2)$ and $(n/2, (n+1)/2)$ containing spin $n+\frac{1}{2}$ by

$$\begin{aligned} F &= \tfrac{1}{4}(2n^2 + 6n + 3) \\ G^{\pm} &= \pm\tfrac{1}{2}(n + \tfrac{3}{2}) \end{aligned} \tag{4.6}$$

we can give $F(p)$ and $G(p)$ in the following form:

$$\begin{aligned} F(p) &= F - 2\sum_{r=1}^{n} \gamma^{\mu_r}\gamma_{\nu_r} - 2\sum_{r\neq p}^{n} g^{\mu_r\mu_p}\gamma_{\nu_r}\gamma_{\nu_p} \\ G(p) &= G^{+}\Gamma^5 - \tfrac{1}{2}\Gamma^5 \sum_{r=1}^{n} \gamma^{\mu_r}\gamma_{\nu_r} \end{aligned} \tag{4.7}$$

where $\Gamma^5 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3$.

The spin operator $S^2(p)$ is as follows

$$S^2(p) = (n + \tfrac{1}{2})(n + \tfrac{3}{2}) + l_1(p) + l_2(p) \tag{4.8}$$

where

$$\begin{aligned} l_1(p) &= -\left\{ \sum_{r=1}^{n} (\gamma^{\mu_r} - \Gamma^0 \varepsilon^{0\mu_r})\gamma_{\nu_r} + \sum_{r\neq p}^{n} (g^{\mu_r\mu_p} - \varepsilon^{0\mu_r}\varepsilon^{0\mu_p})\gamma_{\nu_r}\gamma_{\nu_p} \right\} \\ l_2(p) &= -(2n+1)\sum_{r=1}^{n} \varepsilon^{0\mu_r}\varepsilon^{0}{}_{\nu_r} + \sum_{r\neq p}^{n} g^{\mu_r\mu_p}\varepsilon^{0}{}_{\nu_r}\varepsilon^{0}{}_{\nu_p} + \sum_{r=1}^{n} \gamma^{\mu_r}\varepsilon^{0}{}_{\nu_r}\Gamma^0. \end{aligned} \tag{4.9}$$

Using the relation

$$\gamma_\mu S^2(p)^{\mu\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_n}\psi^{\nu_1\ldots\nu_n} = (n + \tfrac{1}{2})(n + \tfrac{3}{2})\gamma_\mu\psi^{\mu\mu_2\ldots\mu_n} + l[\psi]^{\mu_1\ldots\mu_n}$$

where

$$l[\psi] = -(2n+1)\gamma_\mu \psi^{\mu\mu_2\ldots\mu_n} - 2n\sum_{r\geqslant 2}^{n} \varepsilon^{0\mu_r}\varepsilon^{0}{}_{\nu_r}\gamma_\mu\psi^{\mu\mu_2\ldots\nu_r\ldots\mu_n}$$
$$- \sum_{r\geqslant 2}^{n}(\gamma^{\mu_r} - \Gamma^0\varepsilon^{0\mu_r})\gamma_\mu\gamma_{\nu_r}\psi^{\mu\mu_2\ldots\nu_r\ldots\mu_n} + \sum_{r\geqslant 2}^{n}\gamma^{\mu_r}\Gamma^0\gamma_\mu\varepsilon^{0}{}_{\nu_r}\psi^{\mu\mu_2\ldots\nu_r\ldots\mu_n}$$
$$+ \sum_{r\neq p\geqslant 2}^{n}[g^{\mu_r\mu_p}(\varepsilon^{0}{}_{\nu_r}\varepsilon^{0}{}_{\nu_p}\gamma_\mu\psi^{\mu\mu_2\ldots\nu_r\ldots\nu_p\ldots\mu_n} - \gamma_\mu\gamma_{\nu_r}\gamma_{\nu_p}\psi^{\mu\mu_2\ldots\nu_r\ldots\nu_p\ldots\mu_n})$$
$$+\varepsilon^{0\mu_r}\varepsilon^{0\mu_p}\gamma_\mu\gamma_{\nu_r}\gamma_{\nu_p}\psi^{\mu\mu_2\ldots\nu_r\ldots\nu_p\ldots\mu_n}]$$

one can see that for ψ satisfying

$$S^2(p)\psi = (n+\tfrac{1}{2})(n+\tfrac{3}{2})\psi \tag{4.10}$$

we obtain

$$l[\psi] = 0. \tag{4.11}$$

Equation (4.11) is a homogeneous linear equation for the components of the function $\gamma_\mu\psi^{\mu\mu_2\ldots\mu_n}$. Since the determinant of a given system does not equal zero, we have

$$\gamma_\mu\psi^{\mu\mu_2\ldots\mu_n} = 0.$$

By repeating the reasoning above with p_μ, we find from (4.10)

$$p_\mu\psi^{\mu\mu_2\ldots\mu_n} = 0. \tag{4.12}$$

We once more clarify the role of additional condition (4.2). Using the expressions of $F(p)$, $G(p)$ and $S^2(p)$, we obtain for ψ satisfying (4.2)

$$\begin{aligned} F(p)\psi &= F\psi \\ G(p)\psi &= G^+\Gamma^5\psi \\ S^2(p)\psi &= (n+\tfrac{1}{2})(n+\tfrac{3}{2})\psi + l_2(p)\psi. \end{aligned} \tag{4.13}$$

It is easy to see that the additional condition (4.2) extracts from ψ the components corresponding to the irreducible representations $((n+1)/2, n/2)$ and $(n/2, (n+1)/2)$ of the Lorentz group. From (4.13) it follows that there remain components with lower spins $n-\frac{1}{2},\ldots$. These components are excluded if $l_2(p)\psi = 0$. Using (4.9), one can see that if (4.12) is satisfied, $l_2(p)\psi = 0$. Therefore, conditions (4.2) and (4.12) extract from ψ single spin $n+\frac{1}{2}$.

Here we have condition (4.12) demanding single spin $n+\frac{1}{2}$. In the case of the Rarita–Schwinger equation, (4.12) follows from (4.1) and (4.2), therefore ψ must be the solution of (4.1). In order to demonstrate that the Rarita–Schwinger equation is equivalent to the Bhabha equation we must prove that ψ is an eigenfunction of $S^{05}(p)$.

First we note that when $S^{05}(p)$ acts on the eigenfunctions of the operators $F(p)$, $G(p)$ and $S^2(p)$, the spin remains unchanged, but the functions $\psi_{(k,l)}$ which correspond to some irreducible representation (k, l) of the Lorentz group are transformed to a superposition of functions corresponding to the irreducible representations $(k+\frac{1}{2}, l+\frac{1}{2})$, $(k+\frac{1}{2}, l-\frac{1}{2})$, $(k-\frac{1}{2}, l+\frac{1}{2})$ and $(k-\frac{1}{2}, l-\frac{1}{2})$. In our case we are interested in the representations $((n+1)/2, n/2)$ and $(n/2, (n+1)/2)$ containing the spin $s = n+\frac{1}{2}$. If we take into consideration the irreducible representations present in the representation $(n + \frac{1}{2}, \frac{1}{2})$ of the de Sitter group, we have

$$S^{05}(p)\psi_{((n+1)/2,n/2)} = \alpha\psi_{((n/2,(n+1)/2)} + \beta\psi_{(n/2,(n-1)/2)}$$
$$S^{05}(p)\psi_{(n/2,(n+1)/2)} = \alpha\psi_{((n+1)/2,n/2)} + \beta\psi_{((n-1)/2,n/2)}. \tag{4.14}$$

Since condition (4.2) extracts the irreducible representations $((n+1)/2, n/2)$ and $(n/2, (n+1)/2)$, we must prove the existence of the eigenfunctions of $S^{05}(p)$ which are superpositions of the functions corresponding to these representations. From (4.14) it follows that in this case $\beta = 0$. As we have already mentioned $S^{05}(p)$ commutes with spin. Since the maximum spin in the representations $((n-1)/2, n/2)$ and $(n/2, (n-1)/2)$ is equal to $s = n - \frac{1}{2}$, $\beta = 0$ corresponds to single spin $s = n + \frac{1}{2}$. On the other hand it means that ψ must also satisfy (4.12).

If we denote

$$\psi_1 = \psi_{((n+1)/2,n/2)}(s = n + \tfrac{1}{2}) \qquad \psi_2 = \psi_{(n/2,(n+1)/2)}(s = n + \tfrac{1}{2})$$

then, instead of (4.14), we have

$$S^{05}(p)\psi_1 = \alpha\psi_2 \qquad S^{05}(p)\psi_2 = \alpha\psi_1. \tag{4.15}$$

From (4.13): $G(p)\psi = G^+\Gamma^5\psi$ we obtain

$$\Gamma^5\psi_1 = +\psi_1 \qquad \Gamma^5\psi_2 = -\psi_2.$$

However, $\Gamma^0\Gamma^5 = -\Gamma^5\Gamma^0$, therefore,

$$\Gamma^0\psi_1 = \psi_2 \qquad \Gamma^0\psi_2 = \psi_1. \tag{4.16}$$

From (4.15) and (4.16) it follows that

$$[S^{05}(p), \Gamma^0]\psi_{1,2} = 0 \tag{4.17}$$

therefore one can find common eigenfunctions of the operators $S^{05}(p)$ and Γ^0. Since

$$\Gamma^0 = \varepsilon^0{}_\mu\gamma^\mu = \frac{p_\mu\gamma^\mu}{\sqrt{p^2}}$$

the common eigenfunctions are also the solutions of the Rarita–Schwinger equation (4.1), which finally proves the equivalence of the Rarita–Schwinger and the Bhabha equation for the general spin $s = n + \frac{1}{2}$.

5. Bargmann–Wigner equation

The Bargmann–Wigner equation for a particle with the spin $s = n/2$ has the form (Bargmann and Wigner 1948)

$$\begin{aligned}
&(p_\mu\gamma^\mu - m)_{\alpha_1\beta_1}\psi_{\beta_1\alpha_2\ldots\alpha_n} = 0\\
&(p_\mu\gamma^\mu - m)_{\alpha_2\beta_2}\psi_{\alpha_1\beta_2\ldots\alpha_n} = 0\\
&\vdots\\
&(p_\mu\gamma^\mu - m)_{\alpha_n\beta_n}\psi_{\alpha_1\alpha_2\ldots\beta_n} = 0
\end{aligned} \tag{5.1}$$

where $\psi_{\alpha_1\ldots\alpha_n}$ is symmetrical with respect to the bispinor indices $\alpha_1\ldots\alpha_n$.

In order to demonstrate the relation with the Bhabha equations we rewrite (5.1) as follows

$$\frac{1}{n}\sum_{r=1}^{n} D_r\psi = m\psi \tag{5.2}$$

$$\begin{aligned}(D_1 - D_2)\psi &= 0\\ (D_2 - D_3)\psi &= 0\\ &\vdots\\ (D_{n-1} - D_n)\psi &= 0\end{aligned} \tag{5.3}$$

where

$$D_r = (p_\mu\gamma^\mu)_r. \tag{5.4}$$

In what follows we consider it as equation (5.2) with additional conditions (5.3).

Equation (5.2) is written in the standard form (2.2)

$$(p_\mu\beta^\mu - m)\psi = 0$$

where

$$\beta^\mu = \frac{1}{n}\sum_{r=1}^{n} \gamma_r{}^\mu. \tag{5.5}$$

Now it is easy to verify that the generators

$$S^{\mu 5} = \frac{n}{2}\beta^\mu = \frac{1}{2}\sum_{r=1}^{n} \gamma_r{}^\mu \tag{5.6}$$

generate the de Sitter algebra.

Since (5.2) and (5.3) are equivalent to the original Bargmann–Wigner equation (5.1), and (5.2) is the Bhabha equation, the connection between the Bargmann–Wigner and the Bhabha equation has been proved.

The symmetrical field $\psi_{\alpha_1\ldots\alpha_n}$ corresponds to the irreducible representation $(n/2, n/2)$ of the de Sitter group. From the weight diagram (figure 4) one can see that all $S^{05}(p)$ eigenvalues h describe the spin $n/2$. Only the eigenvalues $h = \pm n/2$ describe a single

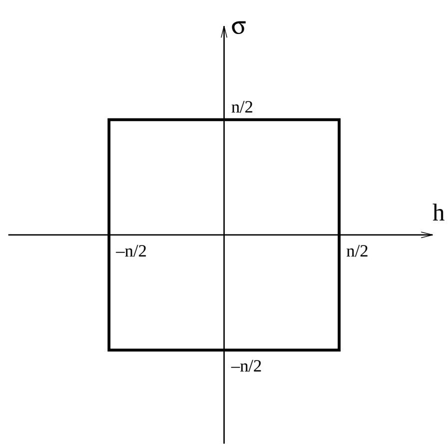

Figure 4. Weight diagram for the irreducible representation $(n/2, n/2)$.

spin, other eigenvalues, $h = \pm(n/2 - 1), \ldots$, describe, in addition to $s = n/2$, also lower spins, $s = n/2 - 1, \ldots$.

Next we analyse the effect of additional conditions (5.3). The Bhabha equation corresponding to (5.2) $S^{05}(p)\psi = h\psi$ is

$$\tfrac{1}{2}\sum_{r=1}^{n} \Gamma^0{}_r \psi = h\psi \tag{5.7}$$

similarly the additional conditions (5.3) are

$$\Gamma^0{}_r \psi = \Gamma^0{}_{r-1}\psi \qquad r = 2, \ldots, n. \tag{5.8}$$

From (5.7) and (5.8) it follows that ψ must be the eigenfunction of all $\Gamma^0{}_r$. The eigenvalues of $\Gamma^0{}_r$ are ± 1. Since ψ is symmetrical with respect to n bispinor indices, all the eigenvalues of $\Gamma^0{}_r$ must also be equal to $+1$ or -1. Therefore, h must be equal to $\pm n/2$, which means that additional conditions (5.8) extract weights $h = \pm n/2$ describing single spin $s = n/2$.

Let us consider a spin. The operator $S^2(p)$ can be written as

$$S^2(p) = \tfrac{3}{8}n(n+1) - \tfrac{1}{8}\sum_{r\neq p}^{n}\sum_{i=1}^{3}(\Gamma^i{}_r \Gamma^i{}_p)^2. \tag{5.9}$$

If we denote the common eigenfunctions of all $\Gamma^0{}_r$ having eigenvalues $+1$ by ψ^+ and, similarly, the eigenfunctions having eigenvalues -1 by ψ^-, we obtain for all r and p

$$\sum_{i=1}^{3}(\Gamma^i{}_r \Gamma^i{}_p)^2 \psi^{\pm} = \psi^{\pm}.$$

Therefore, for these eigenfunctions we obtain from (5.9)

$$S^2(p)\psi^{\pm} = \frac{n}{2}\left(\frac{n}{2}+1\right)\psi^{\pm} \tag{5.10}$$

which again proves that the Bargmann–Wigner equation describes a single spin $s = n/2$.

In conclusion of this section we consider two special cases. The first, corresponding to $n = 1$ is trivial, since it gives us the Dirac equation. The second, corresponding to $n = 2$ is more interesting. Taking $n = 2$, we obtain the Bhabha equation with the matrices

$$\beta^\mu = \tfrac{1}{2}(E \times \gamma^\mu + \gamma^\mu \times E). \tag{5.11}$$

Matrices β^μ satisfy the Kemmer–Duffin relation

$$\beta^\mu \beta^\rho \beta^\nu + \beta^\nu \beta^\rho \beta^\mu = g^{\mu\rho}\beta^\nu + g^{\rho\mu}\beta^\mu. \tag{5.12}$$

Acting on symmetrical bispinor it gives the 10-component Kemmer–Duffin spin 1 equation.

In the $n = 2$ case we had an additional condition

$$(E \times \Gamma^0 - \Gamma^0 \times E)\psi = 0.$$

It appears that on the solutions of the Kemmer–Duffin equation the latter condition is due to the relation

$$(E \times \Gamma^0 - \Gamma^0 \times E)(E \times \Gamma^0 + \Gamma^0 \times E) = 0$$

being automatically fulfilled.

As we have demonstrated, the well known Bargmann–Wigner equation for a particle with spins $s = n/2$ is a Bhabha equation with certain subsidiary conditions.

6. Conclusions

In this paper we have examined the relativistic wave equations connected with the de Sitter group. From the mathematical point of view the de Sitter group and Poincaré group are closely related via contractions and deformations. The Poincaré group is, as is well known, the most important symmetry group in physics. Due to its close relation with the de Sitter group one can expect that the latter group is also playing an important role. The de Sitter group should be treated as an important auxiliary group providing a unified approach to the equations of relativistic quantum field theory.

From the well known wave equations the Dirac and Kemmer–Duffin equations are the Bhabha equations connected with certain irreducible representations of the de Sitter group. In this paper, we demonstrate that two important classes of equations—the Rarita–Schwinger and the Bargmann–Wigner equations—may be treated as the Bhabha equations with certain subsidiary conditions. First we examine the Rarita–Schwinger spin $\frac{3}{2}$ case that corresponds to the representation $(\frac{3}{2}, \frac{1}{2})$. Having the exact form of the corresponding Bhabha equation, we investigate the role of subsidiary conditions in separating one single spin $\frac{3}{2}$. A general proof for the spin $n + \frac{1}{2}$ case is given, which demonstrates that the Rarita–Schwinger equation corresponds to the irreducible representation $(n + \frac{1}{2}, \frac{1}{2})$ of the de Sitter group. We also demonstrate that the Bargmann–Wigner equations correspond to the irreducible representations $(n/2, n/2)$. The role of subsidiary conditions has also been clarified.

The results given here are not, as we hoped, occasional and there should be more equations connected with the de Sitter group. In conclusion, it should be mentioned that the Bhabha equations enable, due to their direct relation with the de Sitter algebra, some important algebraic transformations which are physically important. One of such transformations is, for example, the Foldy–Wouthuysen transformation (Pryce 1948, Foldy and Wouthuysen 1950, Tani 1951, Bracken and Cohen 1969, Loide 1975). For this reason the investigation of the Bhabha equations and their applications in modern field theory is of a certain physical importance, clarifying also the role of the de Sitter group in modern physics.

Acknowledgment

This work was supported in part by the Estonian Science Foundation grant nos 360 and 2275.

References

Antoine J P and Speiser D 1964a *J. Math. Phys.* **5** 1226
——1964b *J. Math. Phys.* **5** 1560
Bargmann V and Wigner E P 1948 *Proc. Natl Acad. Sci.* **34** 211
Bhabha H J 1945 *Rev. Mod. Phys.* **17** 200
Bracken A J and Cohen H A 1969 *J. Math. Phys.* **10** 2024
Corson E M 1952 *Introduction to Tensors, Spinors and Relativistic Wave Equations* (London: Blackie)
Dirac P A M 1936 *Proc. R. Soc.* **A155** 447
Duffin R J 1938 *Phys. Rev.* **54** 1114
Fierz M 1939 *Helv. Phys. Acta* **12** 3
Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29
Gelfand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Group and their Applications* (New York: Pergamon)

Inonü E and Wigner E P 1953 *Proc. Natl Acad. Sci.* **39** 510
Kemmer N 1939 *Proc. R. Soc.* A **173** 91
Krajcik R A and Nieto M M 1974 *Phys. Rev.* D **10** 4049
——1975 *Phys. Rev.* D **11** 1442, 1459
——1976a *Phys. Rev.* D **13** 924
——1976b *Phys. Rev.* D **14** 418
——1977 *Phys. Rev.* D **15** 433, 445
Kõiv M 1967 *Trans. Inst. Phys. Estonian Acad. Sci.* **33** 111
——1969 Irreducible representations of O(5)-group in relativistic particle physics *Preprint* FAI-2 Tartu
Kõiv M, Loide R-K and Meitre J 1970 *Trans. Tallinn Polytechnic Institute* **289** 11
Kõiv M and Saar R 1974 The $O_{1,4}$ type relativistically invariant equation for representation $O_{1,4}[(2n+1)/2, 1/2]$ *Preprint* FI-33 Tartu
Levy-Nahas M 1967 *J. Math. Phys.* **8** 1211
Loide R-K 1972 On some representations of inhomogeneous groups *Preprint* FAI-19 Tartu
——1975 *Teor. Mat. Fiz.* **23** 42
——1984 *J. Phys. A: Math. Gen.* **17** 2535
——1985 *J. Phys. A: Math. Gen.* **18** 2833
Loide R-K, Ots I and Saar R 1992 *Proc. Estonian Acad. Sci. Phys. Math.* **41** 270
——1994 *Proc. Estonian Acad. Sci. Phys. Math.* **43** 25
Lubanski J K 1942a *Physica* **9** 310
——1942b *Physica* **9** 345
Lyakhovsky V 1969 *Commun. Math. Phys.* **14** 70
Pryce M H L 1948 *Proc. R. Soc.* **195A** 62
Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
Saar R, Kõiv M, Ots I and Loide R-K 1993 *J. Math. Phys.* **34** 2866
Saar R, Ots I and Loide R-K 1994 *Hadronic J.* **17** 287
Tani S 1951 *Prog. Theor. Phys.* **6** 267
Velo G and Zwanziger D 1969a *Phys. Rev.* **186** 1337
——1969b *Phys. Rev.* **188** 2218